DISTINGUISHING DERIVED EQUIVALENCE CLASSES USING THE SECOND HOCHSCHILD COHOMOLOGY GROUP

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ABSTRACT. In this paper we study the second Hochschild cohomology group of the preprojective algebra of type D_4 over an algebraically closed field K of characteristic 2. We also calculate the second Hochschild cohomology group of a non-standard algebra which arises as a socle deformation of this preprojective algebra and so show that the two algebras are not derived equivalent. This answers a question raised by Holm and Skowroński.

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INTRODUCTION

The main work of this paper is in determining the second Hochschild cohomology group $\mathrm{HH}^2(\Lambda)$ for two finite dimensional algebras Λ over a field of characteristic 2 in order to show that they are not derived equivalent. We let $\mathcal{A}_1$ denote the preprojective algebra of type D_4; this is a standard algebra. We introduce, in Section 1, the algebra $\mathcal{A}_2$ by quiver and relations; this is a non-standard algebra which is socle equivalent to $\mathcal{A}_1$, in the case where the underlying field has characteristic 2. This work is motivated by the question asked by Holm and Skowroński as to whether or not these two algebras are derived equivalent.

The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are selfinjective algebras of polynomial growth. The main result of this paper (Corollary 4.2) shows that they are not derived equivalent. This answer to the question of Holm and Skowroński enabled them to complete their derived equivalence classification of all symmetric algebras of polynomial growth in [5]. We note that [1] showed that the second Hochschild cohomology group could also be used to distinguish between derived equivalence classes of standard and non-standard algebras of finite representation type.

Throughout this paper, we let Λ denote a finite dimensional algebra over an algebraically closed field K. We start, in Section 1, by giving the quiver and relations for the two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, and recall that we are interested only in the case when $\operatorname{char} K = 2$. (We write our paths in a quiver from left to right.) In Section 2, we give a short description of the projective resolution of [3] which we use to find $\mathrm{HH}^2(\Lambda)$. The remaining two sections determine $\mathrm{HH}^2(\Lambda)$ for $\Lambda = \mathcal{A}_1, \mathcal{A}_2$. As a consequence, we show in Corollary 4.2 that $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$ and hence these two algebras are not derived equivalent.

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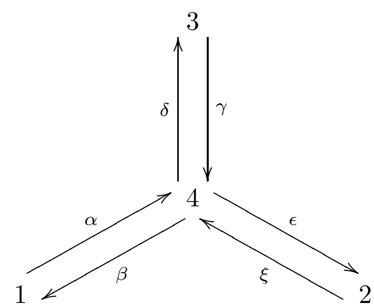
1. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$

In this section we describe the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ by quiver and relations. We assume that K is an algebraically closed field and $\operatorname{char} K = 2$. The standard algebra $\mathcal{A}_1$ is the preprojective algebra of type D_4, and we note that it was shown in [2] that, in the case when $\operatorname{char} K \neq 2$, we have $\mathrm{HH}^2(\mathcal{A}_1) = 0$. We will see that this is in contrast to the $\operatorname{char} K = 2$ case.

The algebra $\mathcal{A}_1$ is the given by the quiver $\mathcal{Q}$:



with relations

$$\beta\alpha + \delta\gamma + \epsilon\xi = 0, \gamma\delta = 0, \xi\epsilon = 0 \text{ and } \alpha\beta = 0.$$

The algebra $\mathcal{A}_2$ is the non-standard algebra given by the same quiver $\mathcal{Q}$ with relations

$$\beta\alpha + \delta\gamma + \epsilon\xi = 0, \gamma\delta = 0, \xi\epsilon = 0, \alpha\beta\alpha = 0, \beta\alpha\beta = 0 \text{ and } \alpha\beta = \alpha\delta\gamma\beta.$$

We need to find a minimal set of generators f^2 for each algebra. We start with the algebra $\mathcal{A}_2$. The set $\{\alpha\beta - \alpha\delta\gamma\beta, \xi\epsilon, \gamma\delta, \beta\alpha + \delta\gamma + \epsilon\xi, \alpha\beta\alpha, \beta\alpha\beta\}$ is not a minimal set of generators for I where $\mathcal{A}_2 = K\mathcal{Q}/I$. Let $x = \beta\alpha + \delta\gamma + \epsilon\xi$ and let $y = \alpha\beta - \alpha\delta\gamma\beta$. We will show that $\alpha\beta\alpha$ is in the ideal generated by $x, y, \gamma\delta, \xi\epsilon$. Using that $\operatorname{char} K = 2$, we have $\alpha\beta\alpha = y\alpha + \alpha\delta\gamma\beta\alpha = y\alpha + \alpha x\beta\alpha + \alpha(\beta\alpha + \epsilon\xi)\beta\alpha = y\alpha + \alpha x\beta\alpha + \alpha\beta\alpha\beta\alpha + \alpha\epsilon\xi x + \alpha\epsilon\xi(\delta\gamma + \epsilon\xi) = y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha\beta\alpha\beta\alpha + \alpha x\delta\gamma + \alpha(\beta\alpha + \delta\gamma)\delta\gamma + \alpha\epsilon\xi\epsilon\xi = y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha x\delta\gamma + \alpha\epsilon\xi\epsilon\xi + \alpha\beta\alpha\beta\alpha + \alpha\beta\alpha x + \alpha\beta\alpha(\beta\alpha + \epsilon\xi) + \alpha\delta\gamma\delta\gamma = y\alpha + \alpha x\beta\alpha + \alpha\epsilon\xi x + \alpha x\delta\gamma + \alpha\epsilon\xi\epsilon\xi + \alpha\beta\alpha x + \alpha\beta\alpha\epsilon\xi + \alpha\delta\gamma\delta\gamma$. However, $\alpha\beta\alpha\epsilon\xi = y\alpha\epsilon\xi + \alpha\delta\gamma\beta\alpha\epsilon\xi = y\alpha\epsilon\xi + \alpha\delta\gamma x\epsilon\xi + \alpha\delta\gamma(\delta\gamma + \epsilon\xi)\epsilon\xi$. Thus $\alpha\beta\alpha$ is in the ideal generated by $x, y, \gamma\delta, \xi\epsilon$. Using a similar argument for $\beta\alpha\beta$, we have that I is generated by the set $\{\alpha\beta - \alpha\delta\gamma\beta, \xi\epsilon, \gamma\delta, \beta\alpha + \delta\gamma + \epsilon\xi\}$. This gives the following result.

Proposition 1.1. *For $\mathcal{A}_2$ let*

$$f_1^2 = \alpha\beta - \alpha\delta\gamma\beta,$$
$$f_2^2 = \xi\epsilon,$$
$$f_3^2 = \gamma\delta,$$
$$f_4^3 = \beta\alpha + \delta\gamma + \epsilon\xi.$$

Then $f^2 = \{f_1^2, f_2^2, f_3^2, f_4^2\}$ is a minimal set of generators of I where $\mathcal{A}_2 = K\mathcal{Q}/I$.

We now consider the algebra $\mathcal{A}_1$.

Proposition 1.2. *For $\mathcal{A}_1$ let*

$$\begin{aligned} f_1^2 &= \alpha\beta, \\ f_2^2 &= \xi\epsilon, \\ f_3^2 &= \gamma\delta, \\ f_4^3 &= \beta\alpha + \delta\gamma + \epsilon\xi. \end{aligned}$$

Then $f^2 = \{f_1^2, f_2^2, f_3^2, f_4^2\}$ is a minimal set of generators for I' where $\mathcal{A}_1 = K\mathcal{Q}/I'$.

2. The Projective resolution

To find the Hochschild cohomology groups for any finite dimensional algebra Λ, a projective resolution of Λ as a Λ,Λ-bimodule is needed. In this section we look at the projective resolutions of [3] and [4] in order to describe the second Hochschild cohomology group. Let K be a field and let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional algebra where $\mathcal{Q}$ is a quiver, and I is an admissible ideal of $K\mathcal{Q}$. Fix a minimal set f^2 of generators for the ideal I. For any $x \in f^2$, we may write $x = \sum_{j=1}^r c_j a_{1j} \cdots a_{kj} \cdots a_{s_j j}$, where the a_{ij} are arrows in $\mathcal{Q}$ and $c_j \in K$, that is, x is a linear combination of paths $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ for $j = 1, \ldots, r$. We may assume that there are (unique) vertices v and w such that each path $a_{1j} \cdots a_{kj} \cdots a_{s_j j}$ starts at v and ends at w for all j, so that $x = vxw$. We write $\mathfrak{o}(x) = v$ and $\mathfrak{t}(x) = w$. Similarly $\mathfrak{o}(a)$ is the origin of the arrow a and $\mathfrak{t}(a)$ is the terminus of a.

In [3, Theorem 2.9], the first 4 terms of a minimal projective resolution of Λ as a Λ,Λ-bimodule are described:

$$\cdots \to Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0.$$

The projective Λ,Λ-bimodules Q^0, Q^1, Q^2 are given by

$$Q^0 = \bigoplus_{v, vertex} \Lambda v \otimes v\Lambda,$$

$$Q^1 = \bigoplus_{a, arrow} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda, \text{ and}$$

$$Q^2 = \bigoplus_{x \in f^2} \Lambda\mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda.$$

Throughout, all tensor products are over K, and we write $\otimes$ for $\otimes_K$. The maps g, A_1, A_2 and A_3 are all Λ,Λ-bimodule homomorphisms. The map $g : Q^0 \to \Lambda$ is the multiplication map so is given by $v \otimes v \mapsto v$. The map $A_1 : Q^1 \to Q^0$ is given by $\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$ for each arrow a. With the notation for $x \in f^2$ given above, the map $A_2 : Q^2 \to Q^1$ is given by $\mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^r c_j (\sum_{k=1}^{s_j} a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j})$, where $a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j} \in \Lambda\mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$.

In order to describe the projective Q^3 and the map A_3 in the Λ,Λ-bimodule resolution of Λ in [3], we need to introduce some notation from [4]. Recall that an element $y \in K\mathcal{Q}$ is uniform if there are vertices v, w such that $y = vy = yw$. We write $\mathfrak{o}(y) = v$ and $\mathfrak{t}(y) = w$. In [4], Green, Solberg and Zacharia show that there are sets f^n in $K\mathcal{Q}$, for $n \geq 3$, consisting of uniform elements $y \in f^n$ such that $y = \sum_{x \in f^{n-1}} x r_x = \sum_{z \in f^{n-2}} z s_z$ for unique elements $r_x, s_z \in K\mathcal{Q}$ such that $s_z \in I$. These sets have special properties related to a minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$, where $\mathfrak{r}$ is the Jacobson radical of Λ. Specifically the n-th projective in the minimal projective Λ-resolution of $\Lambda/\mathfrak{r}$ is $\bigoplus_{y \in f^n} \mathfrak{t}(y)\Lambda$.

In particular, for $y \in f^3$ we may write $y = \sum f_i^2 p_i = \sum q_i f_i^2 r_i$ with $p_i, q_i, r_i \in K\mathcal{Q}$, p_i, q_i in the ideal generated by the arrows of $K\mathcal{Q}$, and p_i unique. Then [3] gives that $Q^3 = \bigoplus_{y \in f^3} \Lambda \mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda$ and, for $y \in f^3$ in the notation above, the component of $A_3(\mathfrak{o}(y) \otimes \mathfrak{t}(y))$ in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 is $\mathfrak{o}(y) \otimes p_i - q_i \otimes r_i$.

Given this part of the minimal projective Λ, Λ-bimodule resolution of Λ

$$Q^3 \xrightarrow{A_3} Q^2 \xrightarrow{A_2} Q^1 \xrightarrow{A_1} Q^0 \xrightarrow{g} \Lambda \to 0$$

we apply $\mathrm{Hom}(-, \Lambda)$ to give the complex

$$0 \to \mathrm{Hom}(Q^0, \Lambda) \xrightarrow{d_1} \mathrm{Hom}(Q^1, \Lambda) \xrightarrow{d_2} \mathrm{Hom}(Q^2, \Lambda) \xrightarrow{d_3} \mathrm{Hom}(Q^3, \Lambda)$$

where d_i is the map induced from A_i for $i = 1, 2, 3$. Then $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3 / \mathrm{Im}\, d_2$.

When considering an element of the projective Λ, Λ-bimodule $Q^1 = \bigoplus_{a, arrow} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ it is important to keep track of the individual summands of Q^1. So to avoid confusion we usually denote an element in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ by $\lambda \otimes_a \lambda'$ using the subscript 'a' to remind us in which summand this element lies. Similarly, an element $\lambda \otimes_{f_i^2} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^2) \otimes \mathfrak{t}(f_i^2)\Lambda$ of Q^2 and an element $\lambda \otimes_{f_i^3} \lambda'$ lies in the summand $\Lambda\mathfrak{o}(f_i^3) \otimes \mathfrak{t}(f_i^3)\Lambda$ of Q^3. We keep this notation for the rest of the paper.

Now we are ready to compute $\mathrm{HH}^2(\Lambda)$ for the finite dimensional algebras $\mathcal{A}_1$ and $\mathcal{A}_2$.

3. $\mathrm{HH}^2(\mathcal{A}_2)$

In this section we determine $\mathrm{HH}^2(\mathcal{A}_2)$ for the non-standard algebra $\mathcal{A}_2$.

Theorem 3.1. *For the non-standard algebra $\mathcal{A}_2$ with $\mathrm{char}\, K = 2$, we have $\dim \mathrm{HH}^2(\mathcal{A}_2) = 4$.*

Proof. The set f^2 of minimal relations was given in Proposition 1.1.

Following [3, 4], we may choose the set f^3 to consist of the following elements:

$$\{f_1^3, f_2^3, f_3^3, f_4^3\}, \text{ where}$$

$$\begin{aligned}
f_1^3 &= f_1^2\alpha\delta\gamma\beta + f_1^2\alpha\beta \\
&= \alpha\delta\gamma\beta f_1^2 + \alpha\beta f_1^2 \in e_1 K\mathcal{Q} e_1, \\
f_2^3 &= f_2^2\xi\delta\gamma\epsilon + f_2^2\xi\beta\alpha\epsilon \\
&= \xi f_4^2\beta\alpha\epsilon + \xi f_4^2\delta\gamma\epsilon + \xi\delta\gamma f_4^2\epsilon + \xi\beta\alpha f_4^2\epsilon + \xi\delta\gamma\epsilon f_2^2 + \xi\beta\alpha\epsilon f_2^2 \in e_2 K\mathcal{Q} e_2, \\
f_3^3 &= f_3^2\gamma\beta\alpha\delta + f_3^2\gamma\epsilon\xi\delta \\
&= \gamma f_4^2\epsilon\xi\delta + \gamma f_4^2\beta\alpha\delta + \gamma\beta\alpha f_4^2\delta + \gamma\epsilon\xi f_4^2\delta + \gamma\beta\alpha\delta f_3^2 + \gamma\epsilon\xi\delta f_3^2 \in e_3 K\mathcal{Q} e_3, \\
f_4^3 &= f_4^2\beta\alpha\delta\gamma + f_4^2\epsilon\xi\delta\gamma \\
&= \epsilon f_2^2\xi\delta\gamma + \delta f_3^2\gamma\beta\alpha + \delta f_3^2\gamma\epsilon\xi + \delta\gamma f_4^2\beta\alpha + \delta\gamma f_4^2\epsilon\xi \\
&+ \beta\alpha f_4^2\delta\gamma + \beta\alpha\delta f_3^2\gamma + \delta\gamma\epsilon\xi f_4^2 + \delta\gamma\beta\alpha f_4^2 \in e_4 K\mathcal{Q} e_4.
\end{aligned}$$

Thus (writing Λ for $\mathcal{A}_2$) the projective $Q^3 = \bigoplus_{y \in f^3} \Lambda\mathfrak{o}(y) \otimes \mathfrak{t}(y)\Lambda = (\Lambda e_1 \otimes e_1\Lambda) \oplus (\Lambda e_2 \otimes e_2\Lambda) \oplus (\Lambda e_3 \otimes e_3\Lambda) \oplus (\Lambda e_4 \otimes e_4\Lambda)$. We know that $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3 / \mathrm{Im}\, d_2$. First we will find $\mathrm{Im}\, d_2$. Let $f \in \mathrm{Hom}(Q^1, \Lambda)$ and so write

$$f(e_1 \otimes_\alpha e_4) = c_1\alpha + c_2\alpha\delta\gamma, \qquad f(e_4 \otimes_\beta e_1) = c_3\beta + c_4\delta\gamma\beta,$$

$$f(e_3 \otimes_\gamma e_4) = c_5\gamma + c_6\gamma\beta\alpha, \qquad f(e_4 \otimes_\delta e_3) = c_7\delta + c_8\beta\alpha\delta,$$

$$f(e_4 \otimes_\epsilon e_2) = c_9\epsilon + c_{10}\delta\gamma\epsilon \qquad \text{and } f(e_2 \otimes_\xi e_4) = c_{11}\xi + c_{12}\xi\delta\gamma,$$

where $c_1, c_2, c_3, c_4, \ldots, c_{12} \in K$. Now we find $fA_2 = d_2 f$. We have $fA_2(e_1 \otimes_{f_1^2} e_1) = f(e_1 \otimes_\alpha e_4)\beta + \alpha f(e_4 \otimes_\beta e_1) - f(e_1 \otimes_\alpha e_4)\delta\gamma\beta - \alpha f(e_4 \otimes_\delta e_3)\gamma\beta - \alpha\delta f(e_3 \otimes_\gamma e_4)\beta - \alpha\delta\gamma f(e_4 \otimes_\beta e_1) = c_1\alpha\beta + c_2\alpha\delta\gamma\beta + c_3\alpha\beta + c_4\alpha\delta\gamma\beta - c_1\alpha\delta\gamma\beta - c_7\alpha\delta\gamma\beta - c_5\alpha\delta\gamma\beta - c_3\alpha\delta\gamma\beta = (c_1 + c_2 + c_3 + c_4 - c_1 - c_7 - c_5 - c_3)\alpha\beta = (c_2 + c_4 + c_7 + c_5)\alpha\beta$.

Also $fA_2(e_2 \otimes_{f_2^2} e_2) = f(e_2 \otimes_\xi e_4)\epsilon + \xi f(e_4 \otimes_\epsilon e_2) = (c_{12} + c_{10})\xi\delta\gamma\epsilon$.

We have $fA_2(e_3 \otimes_{f_3^2} e_3) = f(e_3 \otimes_\gamma e_4)\delta + \gamma f(e_4 \otimes_\delta e_3) = (c_6 + c_8)\gamma\beta\alpha\delta$.

And $fA_2(e_4 \otimes_{f_4^2} e_4) = f(e_4 \otimes_\beta e_1)\alpha + f(e_4 \otimes_\delta e_3)\gamma + f(e_2 \otimes_\epsilon e_4)\xi + \beta f(e_1 \otimes_\alpha e_4) + \delta f(e_3 \otimes_\gamma e_4) + \epsilon f(e_2 \otimes_\xi e_4) = c_3\beta\alpha + c_4\delta\gamma\beta\alpha + c_7\delta\gamma + c_8\beta\alpha\delta\gamma + c_9\epsilon\xi + c_{10}\delta\gamma\epsilon\xi + c_1\beta\alpha + c_2\beta\alpha\delta\gamma + c_5\delta\gamma + c_6\delta\gamma\beta\alpha + c_{11}\epsilon\xi + c_{12}\epsilon\xi\delta\gamma = (c_3 + c_1)\beta\alpha + (c_7 + c_5)\delta\gamma + (c_9 + c_{11})\epsilon\xi + (c_4 + c_2 + c_7 + c_5 + c_{10} + c_{12})\delta\gamma\beta\alpha = (c_3 + c_1 + c_9 + c_{11})\beta\alpha + (c_7 + c_5 + c_9 + c_{11})\delta\gamma + (c_4 + c_2 + c_7 + c_5 + c_{10} + c_{12})\delta\gamma\beta\alpha$.

Hence, fA_2 is given by

$$fA_2(e_1 \otimes_{f_1^2} e_1) = d_1\alpha\beta,$$

$$fA_2(e_2 \otimes_{f_2^2} e_2) = d_2\xi\delta\gamma\epsilon,$$

$$fA_2(e_3 \otimes_{f_3^2} e_3) = d_3\gamma\beta\alpha\delta,$$

$$fA_2(e_4 \otimes_{f_4^2} e_4) = d_4\beta\alpha + d_5\delta\gamma + (d_1 + d_2)\delta\gamma\beta\alpha,$$

for some $d_1, \ldots, d_5 \in K$. So $\dim \operatorname{Im} d_2 = 5$.

Now we determine $\operatorname{Ker} d_3$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Let $h : Q^2 \to \Lambda$ be given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2\alpha\delta\gamma\beta,$$

$$h(e_2 \otimes_{f_2^2} e_2) = c_3 e_2 + c_4\xi\delta\gamma\epsilon,$$

$$h(e_3 \otimes_{f_3^2} e_3) = c_5 e_3 + c_6\gamma\beta\alpha\delta \text{ and}$$

$$h(e_4 \otimes_{f_4^2} e_4) = c_7 e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma,$$

for some $c_1, c_2, \ldots, c_{10} \in K$.

Then $hA_3(e_1 \otimes_{f_1^3} e_1) = h(e_1 \otimes_{f_1^2} e_1)\alpha\delta\gamma\beta + h(e_1 \otimes_{f_1^2} e_1)\alpha\beta - \alpha\delta\gamma\beta h(e_1 \otimes_{f_1^2} e_1) - \alpha\beta h(e_1 \otimes_{f_1^2} e_1) = c_1\alpha\delta\gamma\beta + c_1\alpha\beta - c_1\alpha\delta\gamma\beta - c_1\alpha\beta = 0$,

In a similar way and recalling that $\operatorname{char} K = 2$, we can show that $hA_3(e_2 \otimes_{f_2^3} e_2) = 0$ and $hA_3(e_3 \otimes_{f_3^3} e_3) = 0$.

Finally, $hA_3(e_2 \otimes_{f_4^3} e_2) = h(e_4 \otimes_{f_4^2} e_4)\beta\alpha\delta\gamma + h(e_4 \otimes_{f_4^2} e_4)\epsilon\xi\delta\gamma - \epsilon h(e_2 \otimes_{f_2^2} e_2)\xi\delta\gamma - \delta h(e_3 \otimes_{f_3^2} e_3)\gamma\beta\alpha - \delta h(e_3 \otimes_{f_3^2} e_3)\gamma\epsilon\xi - \delta\gamma h(e_4 \otimes_{f_4^2} e_4)\beta\alpha - \delta\gamma h(e_4 \otimes_{f_4^2} e_4)\epsilon\xi - \beta\alpha h(e_4 \otimes_{f_4^2} e_4)\delta\gamma - \beta\alpha\delta h(e_3 \otimes_{f_3^2} e_3)\gamma - \delta\gamma\epsilon\xi h(e_4 \otimes_{f_4^2} e_4) - \delta\gamma\beta\alpha h(e_4 \otimes_{f_4^2} e_4) = c_7\beta\alpha\delta\gamma + c_7\epsilon\xi\delta\gamma - c_3\epsilon\xi\delta\gamma - c_5\delta\gamma\beta\alpha - c_5\delta\gamma\epsilon\xi - c_7\delta\gamma\beta\alpha - c_7\delta\gamma\epsilon\xi - c_7\beta\alpha\delta\gamma - c_5\delta\gamma\beta\alpha - c_7\delta\gamma\epsilon\xi - c_7\delta\gamma\beta\alpha = (c_7 - c_3 - c_5)\epsilon\xi\delta\gamma$. As $h \in \operatorname{Ker} d_3$ we have $c_7 = c_3 + c_5$.

Thus h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1 e_1 + c_2\alpha\delta\gamma\beta,$$

$$h(e_2 \otimes_{f_2^2} e_2) = c_3 e_2 + c_4\xi\delta\gamma\epsilon,$$

$$h(e_3 \otimes_{f_3^2} e_3) = c_5 e_3 + c_6\gamma\beta\alpha\delta \text{ and}$$

$$h(e_4 \otimes_{f_4^2} e_4) = (c_3 + c_5)e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma.$$

Hence $\dim \operatorname{Ker} d_3 = 9$.

Therefore, $\dim \operatorname{HH}^2(\mathcal{A}_2) = \dim \operatorname{Ker} d_3 - \dim \operatorname{Im} d_2 = 9 - 5 = 4$. $\square$

4. $\mathrm{HH}^2(\mathcal{A}_1)$

In this section we determine $\mathrm{HH}^2(\mathcal{A}_1)$ for the standard algebra $\mathcal{A}_1$.

Theorem 4.1. *For the standard algebra* $\mathcal{A}_1$ *with* $\operatorname{char} K = 2$*, we have* $\dim \mathrm{HH}^2(\mathcal{A}_1) = 3$.

Proof. The set f^2 of minimal relations was given in Proposition 1.2. Following [3, 4], we may choose the set f^3 to consist of the following elements:

$$\{f_1^3, f_2^3, f_3^3, f_4^3\}, \text{ where}$$

$$\begin{array}{lllllllllllll}
f_1^3 & = & f_1^2\alpha\epsilon\xi\beta \\
 & = & \alpha f_4^2\epsilon\xi\beta & + & \alpha\delta\gamma f_4^2\beta & + & \alpha\delta\gamma\beta f_1^2 & + & \alpha\delta f_3^2\gamma\beta & + & \alpha\epsilon f_2^2\xi\beta & \in & e_1K\mathcal{Q}e_1, \\
f_2^3 & = & f_2^2\xi\delta\gamma\epsilon \\
 & = & \xi f_4^2\delta\gamma\epsilon & + & \xi\beta\alpha f_4^2\epsilon & + & \xi\beta f_1^2\alpha\epsilon & + & \xi\beta\alpha\epsilon f_2^2 & + & \xi\delta f_3^2\gamma\epsilon & \in & e_2K\mathcal{Q}e_2, \\
f_3^3 & = & f_3^2\gamma\epsilon\xi\delta \\
 & = & \gamma f_4^2\epsilon\xi\delta & + & \gamma\beta\alpha f_4^2\delta & + & \gamma\beta f_1^2\alpha\delta & + & \gamma\beta\alpha\delta f_3^2 & + & \gamma\epsilon f_2^2\xi\delta & \in & e_3K\mathcal{Q}e_3, \\
f_4^3 & = & f_4^2\beta\alpha\delta\gamma \\
 & = & \beta f_1^2\alpha\delta\gamma & + & \delta f_3^2\gamma\epsilon\xi & + & \epsilon f_2^2\xi\delta\gamma & + & \delta\gamma f_4^2\epsilon\xi & + & \epsilon\xi f_4^2\delta\gamma \\
 & + & \delta\gamma\beta f_1^2\alpha & + & \delta\gamma\epsilon f_2^2\xi & + & \epsilon\xi\delta f_3^2\gamma & + & \delta\gamma\beta\alpha f_4^2 & & & \in & e_4K\mathcal{Q}e_4.
\end{array}$$

Thus (writing Λ for $\mathcal{A}_1$) the projective $Q^3 = \bigoplus_{y\in f^3} \Lambda\mathfrak{o}(y)\otimes\mathfrak{t}(y)\Lambda = (\Lambda e_1\otimes e_1\Lambda)\oplus(\Lambda e_2\otimes e_2\Lambda)\oplus(\Lambda e_3\otimes e_3\Lambda)\oplus(\Lambda e_4\otimes e_4\Lambda)$.

Again, $\mathrm{HH}^2(\Lambda) = \operatorname{Ker} d_3/\operatorname{Im} d_2$. First we will find $\operatorname{Im} d_2$. Let $f \in \operatorname{Hom}(Q^1, \Lambda)$ and so write

$$f(e_1\otimes_\alpha e_4) = c_1\alpha + c_2\alpha\delta\gamma, \qquad f(e_4\otimes_\beta e_1) = c_3\beta + c_4\delta\gamma\beta,$$
$$f(e_3\otimes_\gamma e_4) = c_5\gamma + c_6\gamma\beta\alpha, \qquad f(e_4\otimes_\delta e_3) = c_7\delta + c_8\beta\alpha\delta,$$
$$f(e_4\otimes_\epsilon e_2) = c_9\epsilon + c_{10}\delta\gamma\epsilon \qquad \text{and } f(e_2\otimes_\xi e_4) = c_{11}\xi + c_{12}\xi\delta\gamma,$$

where $c_1, c_2, c_3, c_4, \ldots, c_{12} \in K$. Now we find $fA_2 = d_2 f$. We have $fA_2(e_1\otimes_{f_1^2} e_1) = f(e_1\otimes_\alpha e_4)\beta + \alpha f(e_4\otimes_\beta e_1) = c_2\alpha\delta\gamma\beta + c_4\alpha\delta\gamma\beta = (c_2+c_4)\alpha\delta\gamma\beta$.

Also $fA_2(e_2\otimes_{f_2^2} e_2) = f(e_2\otimes_\xi e_4)\epsilon + \xi f(e_4\otimes_\epsilon e_2) = (c_{12}+c_{10})\xi\delta\gamma\epsilon$ and $fA_2(e_3\otimes_{f_3^2} e_3) = f(e_3\otimes_\gamma e_4)\delta + \gamma f(e_4\otimes_\delta e_3) = (c_6+c_8)\gamma\beta\alpha\delta$.

And $fA_2(e_4\otimes_{f_4^2} e_4) = f(e_4\otimes_\beta e_1)\alpha + f(e_4\otimes_\delta e_3)\gamma + f(e_2\otimes_\epsilon e_4)\xi + \beta f(e_1\otimes_\alpha e_4) + \delta f(e_3\otimes_\gamma e_4) + \epsilon f(e_2\otimes_\xi e_4) = (c_3+c_9+c_1+c_{11})\beta\alpha + (c_7+c_9+c_5+c_{11})\delta\gamma + (c_4+c_8+c_{10}+c_2+c_6+c_{12})\delta\gamma\beta\alpha$. Hence, fA_2 is given by

$$fA_2(e_1\otimes_{f_1^2} e_1) = d_1\alpha\beta,$$
$$fA_2(e_2\otimes_{f_2^2} e_2) = d_2\xi\delta\gamma\epsilon,$$
$$fA_2(e_3\otimes_{f_3^2} e_3) = d_3\gamma\beta\alpha\delta,$$
$$fA_2(e_4\otimes_{f_4^2} e_4) = d_4\beta\alpha + d_5\delta\gamma + (d_1+d_2+d_3)\delta\gamma\beta\alpha,$$

for some $d_1, \ldots, d_5 \in K$. So $\dim \operatorname{Im} d_2 = 5$.

Now we determine $\operatorname{Ker} d_3$. Let $h \in \operatorname{Ker} d_3$, so $h \in \operatorname{Hom}(Q^2, \Lambda)$ and $d_3 h = 0$. Let $h : Q^2 \to \Lambda$ be given by

$$h(e_1\otimes_{f_1^2} e_1) = c_1 e_1 + c_2\alpha\delta\gamma\beta,$$
$$h(e_2\otimes_{f_2^2} e_2) = c_3 e_2 + c_4\xi\delta\gamma\epsilon,$$
$$h(e_3\otimes_{f_3^2} e_3) = c_5 e_3 + c_6\gamma\beta\alpha\delta \text{ and}$$
$$h(e_4\otimes_{f_4^2} e_4) = c_7 e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma,$$

for some $c_1, c_2, \ldots, c_{10} \in K$.

It can be easily shown that $hA_3(e_1 \otimes_{f_1^3} e_1) = (-c_5 - c_3)\alpha\delta\gamma\beta$. As $h \in \operatorname{Ker} d_3$ and $\operatorname{char} K = 2$ we have $c_5 = c_3$, and that $hA_3(e_2 \otimes_{f_2^3} e_2) = (-c_1 - c_5)\xi\delta\gamma\epsilon$ so that $c_1 = c_5$. Similarly, $hA_3(e_3 \otimes_{f_3^3} e_3) = (-c_1 - c_3)\gamma\beta\alpha\delta$ so that $c_1 = c_3$. Finally, we have $hA_3(e_2 \otimes_{f_4^3} e_2) = 0$.

Thus h is given by

$$h(e_1 \otimes_{f_1^2} e_1) = c_1e_1 + c_2\alpha\delta\gamma\beta,$$
$$h(e_2 \otimes_{f_2^2} e_2) = c_1e_2 + c_4\xi\delta\gamma\epsilon,$$
$$h(e_3 \otimes_{f_3^2} e_3) = c_1e_3 + c_6\gamma\beta\alpha\delta \text{ and}$$
$$h(e_4 \otimes_{f_4^2} e_4) = c_7e_4 + c_8\beta\alpha + c_9\delta\gamma + c_{10}\beta\alpha\delta\gamma.$$

Hence dim Ker $d_3 = 8$.

Therefore dim $\mathrm{HH}^2(\mathcal{A}_1) = $ dim Ker $d_3 -$ dim Im $d_2 = 8 - 5 = 3$. □

Thus we have shown that $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$. Hence these two algebras are not derived equivalent. Now we state the main result of this paper.

Corollary 4.2. *For the finite dimensional algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over an algebraically closed field K with* $\operatorname{char} K = 2$*, we have* $\dim \mathrm{HH}^2(\mathcal{A}_1) \neq \dim \mathrm{HH}^2(\mathcal{A}_2)$*. Hence these two algebras are not derived equivalent.*

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